A Hybrid Method for Annular Plate Bending Problems

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ABSTRACT

A hybrid method, which couples equilibrium equations and compatibility conditions developed based on the equilibrium equations, is proposed for the analysis of annular plate bending problems. An axisymmetric line element having 2 force and 4 displacement degrees of freedom is developed with the necessary matrix formulation based on the integrated force method. In addition to convergence study, a variety of problems of annular plates under different loading and boundary condition are solved. Matlab software is exploited to plot moment contours and deformed geometry of annular plate. Results are compared with the available classical solutions to demonstrate the effectiveness of the proposed method; a good agreement is indicated.

Keywords - **Axisymmetric element, Bending problems, Convergence study, Integrated force method, Matlab software**

1. INTRODUCTION

Annular plate is a plane structural geometry with circular plan having a circular hole at the center with specific purpose with dimensions as per practical applications, i.e. pressure vessels, automotive and large machinery components etc. If the thickness of circular plate is not greater than one tenth of the diameter, one needs not to model it as a 3D continuum and simple 2D plate theory can be applied to calculate the deformation and stresses. Further, if the plate is axisymmetrically loaded instead of asymmetrically loaded, its axisymmetric behavior allows one to treat it as a 1D problem.

The governing equations for circular plated structures can be obtained by using either simple mathematical transformation from rectangular cartesian coordinate system to polar coordinate system or by direct derivation in the polar coordinate system [1]. Each approach has its own mathematical insights and loopholes. As transformation from rectangular to polar coordinate system needs quite a bit of mathematical manipulations, it is generally not preferred.

There exist various numerical methods to solve the problems of annular plate such as Finite Difference Method (FDM), Boundary Element Method (BEM), Finite Strip Method (FSM) and Finite Element Method (FEM) [2]. Few direct and indirect approaches are also available, where it bypasses the mathematical derivation of the Euler-Langrage's equations and start directly from the variation formulation. One such direct method was proposed by Lord Rayleigh and then it was generalized independently by Ritz; later on it was named as Rayleigh-Ritz method [1]. However, the Rayleigh-Ritz method for annular plate does not give the exact solution due to lack of logarithmic terms.

In the present paper, a hybrid method for handling the axisymmetric plate bending problems with circular hole at the center is considered. The method is based on Integrated Force Method (IFM) [3] which combines the Equilibrium Equations (EEs) and the Compatibility Conditions (CCs) developed based on EEs by using a systematic concatenation procedure. By using this approach, one can calculate the internal moments and then the nodal displacements; if required. Recent development in integrated force method comprises of formulation of new novel condition, which completes the Beltrami-Michell's Formulation (BMF) in polar coordinates, which have been validated in past by solving mixed Boundary Value Problem (BVP) [4 -5].

A two noded radial element is developed in the present work with two force and four displacement degrees of freedom. Necessary matrices are derived by discretizing the expressions for potential and complimentary strain energies. After describing the solution procedure, a variety of annular plate problems are analysed under Uniform Lateral Pressure (ULP) and Line Loading (LL) acting along either outer edge or inner edge. Results obtained for internal moments and deflections are compared with the available analytical solutions [6].

2. THE MATRIX FORMULATION

2.1 The Hybridization Concept

The equations for a continuum discretized into finite number of elements with 'n' and 'm' force and displacement degrees of freedom respectively, are obtained in the hybrid method by coupling the 'm` number of equilibrium equations and r = n – m compatibility conditions. The m equilibrium equations (EE) are written as:

$$[B]\{F\} = \{P\} \quad \ldots (1)$$

and the' r' compatibility conditions are written as:

$$[C][G]\{F\} = \{\delta R\} \quad \ldots (2)$$

The governing equations are expressed as:

$$\begin{bmatrix} [B] \\ [C][G] \end{bmatrix}\{F\} = \begin{bmatrix} \{P\} \\ \{\delta R\} \end{bmatrix} \text{ Or } [S]\{F\} = \{P\} \quad \ldots (3)$$

Displacements {X} are calculated using the following equation:

$$\{X\} = [J]\{[G]\{F\} + \{\beta^0\}\} \quad \ldots (4)$$

In the above equation, {F} is obtained using (3), [J] = m rows of $[[S]^{-1}]^T$, [B] is equilibrium matrix of m x n size which is sparse and unsymmetrical, [G] is a symmetrical flexibility matrix; it is a block-diagonal matrix, where each block represents the element flexibility matrix for an element, [C] is the compatibility matrix of size r x n, {δR} = - [C] $\{\beta\}^0$ is the effective deformation vector with $\{\beta\}^0$ being the initial deformation vector of dimension 'n', [S] is the IFM governing unsymmetrical matrix of size n x n, [J] is the m x n size deformation coefficient matrix which is back-calculated from [S] matrix.

2.2 The Equilibrium Matrix [Be]

The elemental equilibrium matrix written in terms of forces at nodal points represents the vectorial summation of 'n' internal forces {F} and 'm' external loads {P}. The nodal EE in matrix notation can be stored as rectangular matrix $[B_e]$ of size m x n. The variation functional is evaluated as a portion of IFM functional which yields the basic elemental equilibrium matrix $[B_e]$ in explicit form as follows:

$$U_p = \int_D \left\{ M_r \frac{\partial^2 w}{\partial r^2} + M_\theta \frac{1}{r}\frac{\partial w}{\partial r} \right\} 2\pi r\, dr \quad \ldots (5)$$

$$= \int_D \{M\}^T \{\varepsilon\} da \quad \ldots (6)$$

where $\{M\}^T = (M_r, M_\theta)$ are the internal moments and $\{\epsilon\}^T = \left(\frac{\partial^2 w}{\partial r^2}, \frac{1}{r}\frac{\partial w}{\partial r} \right)$ represents the curvatures corresponding to each internal moment.

Consider a two-noded, 4 displacement degrees of freedom (X_1 to X_4) line element of thickness t with length as 'a' along the radial direction and r_1 and r_2 as radius of left and right nodes respectively as shown in Fig.1.

The force field is chosen in terms of two independent forces F_1 and F_2. Relations between internal moments and independent forces are written in matrix form as:

$$\begin{Bmatrix} M_r \\ M_\theta \end{Bmatrix} = \begin{bmatrix} 1 & r \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\text{or } \{M\} = [Y]\{F_e\} \quad \ldots (7)$$

Displacement function by considering r_1 and r_2 as radius of extreme points of the element can be written as:

$$w(r,\theta) = N_1X_1 + N_2X_2 + N_3X_3 + N_4X_4$$

$$= [N]\{X\} \quad \ldots (8)$$

Where $N_1 = \left(\frac{r - r_2}{r_1 - r_2}\right)^2 \left[1 + 2\left(\frac{r_1 - r}{r_1 - r_2}\right)\right]$,

$N_2 = \left(\frac{r - r_2}{r_1 - r_2}\right)^2 (r - r_1)$,

$N_3 = \left(\frac{r - r_1}{r_2 - r_1}\right)^2 \left[1 + 2\left(\frac{r_2 - r}{r_2 - r_1}\right)\right]$ and

$$N_4 = \left(\frac{r - r_1}{r_2 - r_1}\right)^2 (r - r_2)$$

where [N] is a shape function matrix which is a function of r and {X} is a displacement vector. By arranging all force and displacement functions properly, one can discretize the (5) to obtain the elemental equilibrium matrix as follows:

$$U^e = \{X\}^T [B_e] \{F\} \quad \ldots (9)$$

$$\text{where } [B_e] = 2\pi \int_{r_1}^{r_2} [Z]^T [Y]\, r\, dr \quad \ldots (10)$$

where again [Z] = [L][N]; [L] is the differential operator matrix with respect to r, [N] is the displacement interpolation function matrix and [Y] is the force interpolation function matrix. After calculating [Z] matrix and stress interpolation matrix [Y] and integrating, the equilibrium matrix $[B_e]$ is obtained using the relation given in (10). For element 1, for example, with r_1 = b and r_2 = (a + b), the $[B_e]$ matrix will be as follows:

$$[B_e] = \begin{bmatrix} 0 & \frac{b}{2} - a \\ 0 - \frac{b^2}{3} & -\frac{a^2}{6} \\ 0 & a + \frac{b}{2} \\ -a & -\frac{5a^2}{6} \end{bmatrix} \quad \ldots (11)$$

2.3 The Flexibility Matrix $[G_e]$

The basic elemental flexibility matrix is obtained by discretizing the complementary strain energy which gives:

$$[G_e] = 2\pi \int_{r_1}^{r_2} [Y]^T [D] [Y]\, r\, dr \quad \ldots (12)$$

where [Y] is moment interpolation function matrix and [D] is material property matrix of size 2 x 2. Substituting values in (12) and integrating, it yields the symmetrical flexibility matrix $[G_e]$ which for the first element is as follows:

$$[G_e] = \frac{24\pi}{Et^3} \begin{bmatrix} \frac{7}{10}a^2 & \frac{7}{30}a^3 + \frac{14}{15}a^2 b \\ \frac{7}{30}a^3 + \frac{14}{15}a^2 b & a^4 + \frac{a^7 b^7}{8} \end{bmatrix} \quad \ldots (13)$$

2.4 The Compatibility Matrix [C]

The compatibility matrix is obtained from the deformation displacement relation ($\{\beta\} = [B]^T\{X\}$). In DDR all the deformations are expressed in terms of all possible nodal displacements and the 'n - m' compatibility conditions are developed in terms of internal forces i.e., F_1,------Fn where 'n' is the total number of internal forces in a given problem. The concatenating or global compatibility matrix [C] can be evaluated by multiplying the coefficients of the compatibility conditions and the global flexibility matrix [G].

3. THE SOLUTION PROCEDURE

Due to axi-symmetric nature of the problem, only one radial line is considered for the study of behavior of annular plate. A two noded line element having length 'a' with two force and four displacement degrees of freedom is used for discretizing the problem into desired number of elements. The $[B_e]$ matrix is obtained by substituting the values of length of element 'a' and internal radius of annular plate 'b' in (11), which gives a global equilibrium matrix. The load vector {P} is then calculated by using (3) depending upon the type of loading on the plate. Primary unknowns, the forces {F} are then obtained by using Matlab's inverting routine [7]. Finally, the displacements are obtained by using the relation ({X} = [J] [G] {F}), where [J] = m rows of matrix $[[S]^{-1}]^T$.

Software is developed for annular plate bending problems using VB.NET programming language in which different forms are developed for input of data using different text boxes. Automated link is also developed between Matlab based auto-generation CC module and input data. Transferring of data of various necessary matrices is also automated, which is needed for the further calculation. Different plots for the plate such as deformed shape and moment contours are managed through Matlab software.

4. ANNULAR PLATE BENDING EXAMPLES

The data assumed for numerical study of axisymmetrically loaded annular plates is as follows. A Uniform Lateral Pressure (ULP) q of 10^6 N/m^2 (1 MPa) on total plate surface and Line Loading (LL) p of 1000 N/m at inner or outer edge of the annular plate are considered. The outer and inner diameters of the plate are taken as 200 mm and 80 mm respectively with thickness of plate as 10 mm. The modulus of elasticity of steel plate is considered as 2.01 x 10^{11} N/m^2 with Poisson's ratio as 0.3 [6].

Using the developed software, first, a convergence study is carried out to decide the suitable number of elements. Fig. 2 shows the convergence graph for deflection at the outer edge for annular plate with clamped condition at the inner edge and free at the outer boundary subjected to uniformly lateral pressure of 1 MPa. As can be seen from the convergence graph, discretization with 5 elements gives results quite close to the exact solution and therefore for remaining examples 5 element discretization is used along a radial line. In Table 1 results obtained for deflection w and moments Mr and M_Θ along a radial line at different nodal points are compared with the available solution [6] for uniform lateral pressure q and line loading p for clamped boundary conditions. Figs. 3 and 4 depict the contours for Mr and M_Θ of right upper part of quarter annular plate for plate subjected to line loading and having clamped boundary condition at inner edge. Fig. 5 shows the complete 3D deformed shape for a clamped outer periphery plate when inner edge is subjected to LL.

Next, an example of an annular plate subjected to uniform lateral pressure with support conditions as simply supported at inner and outer edges is considered. The results obtained for lateral deflection w and moments Mr and M_Θ at internal nodes are tabulated in Table 2 whereas Fig. 6 depicts the deformed shape.

The last example considered here is that of an annular plate having inner edge as clamped and outer edges as simply supported. The results obtained under uniform lateral pressure are reported in Table 2.

5. CONCLUSIONS

The proposed hybrid method which is based on integration of equilibrium and compatibility conditions is found quite suitable for handling axisymmetrically loaded annular plate bending problems. The method is found to provide lower bound solution for lateral deflection with monotonic convergence towards the exact solution.

The comparison of results obtained for deflections and moments using the developed code based on the suggested hybrid method with 5 element discretization indicated a good agreement with those based on the classical methods [6] for a variety of problems under different boundary conditions at inner and outer periphery when the plate is subjected to uniform lateral pressure or a line load along inner or outer periphery.

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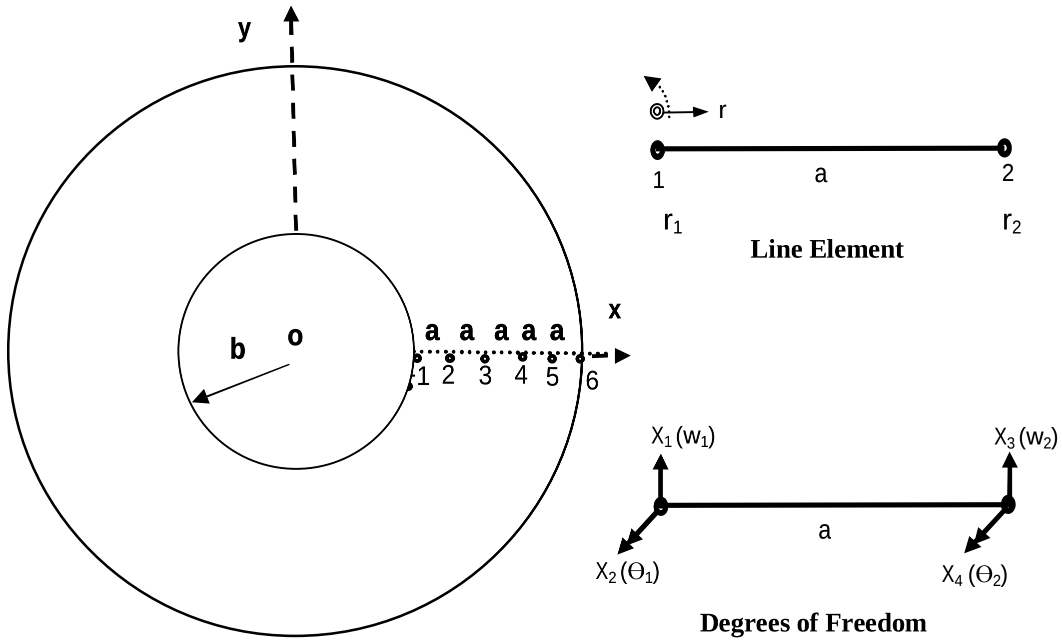


Fig. 1 Annular plate showing discretization including element DOF

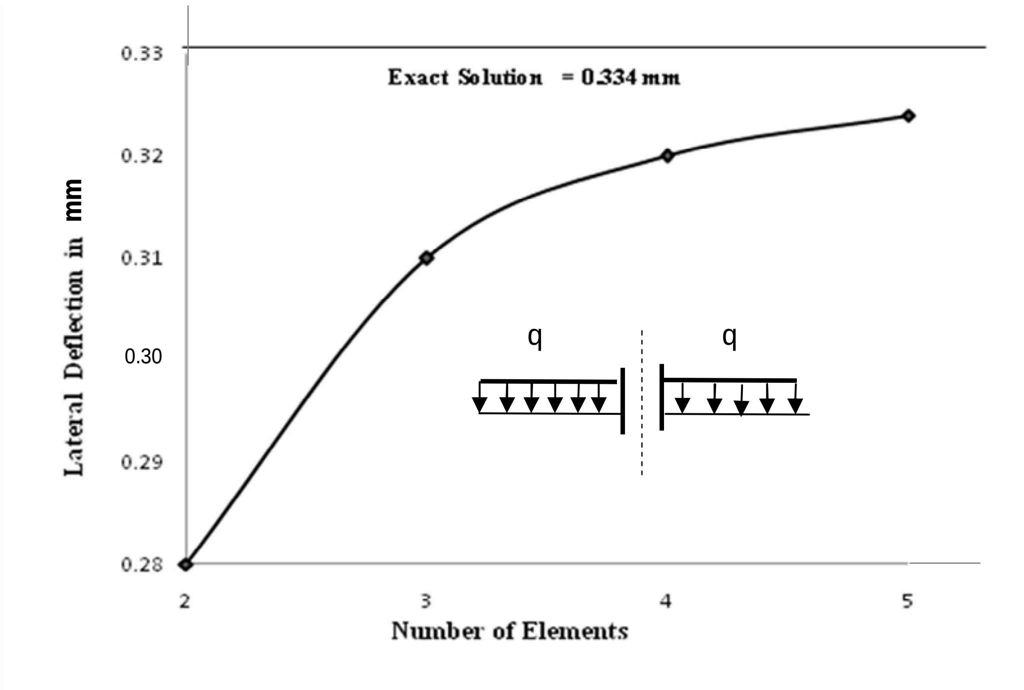


Fig. 2 Convergence graph for deflection at outer periphery

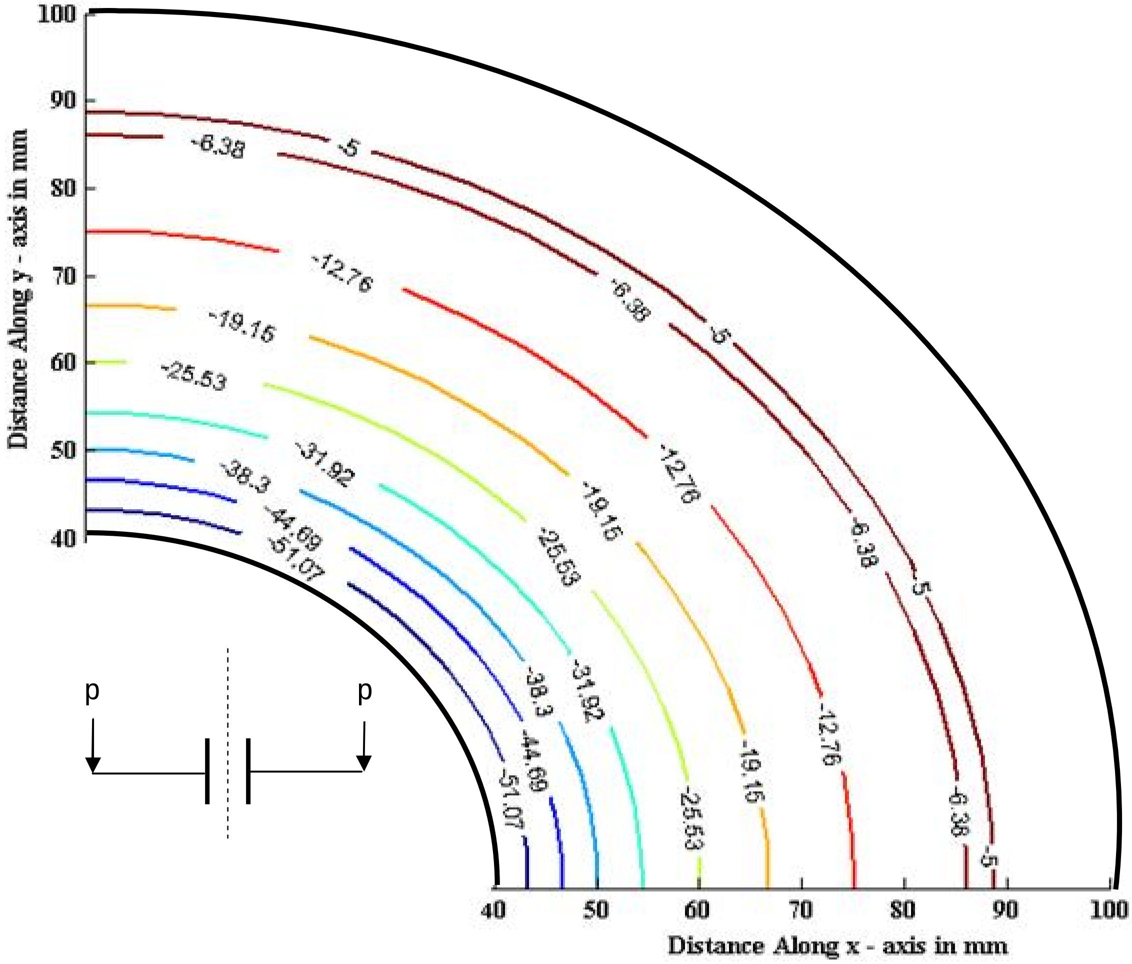


Fig. 3 Contours of Mr for plate clamped along inner edge and outer edge under line loading

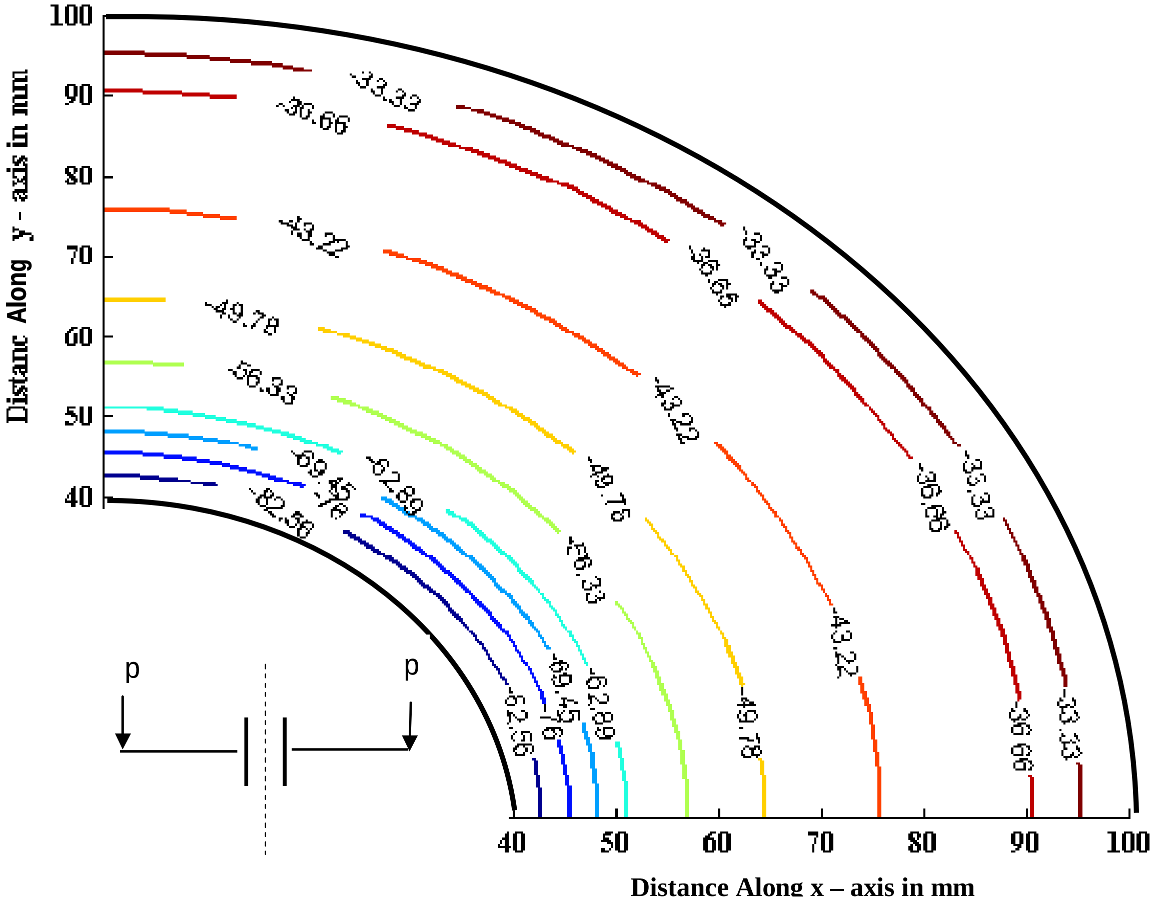


Fig. 4 Contours of M_0 for plate clamped along inner edge and outer edge under line loading

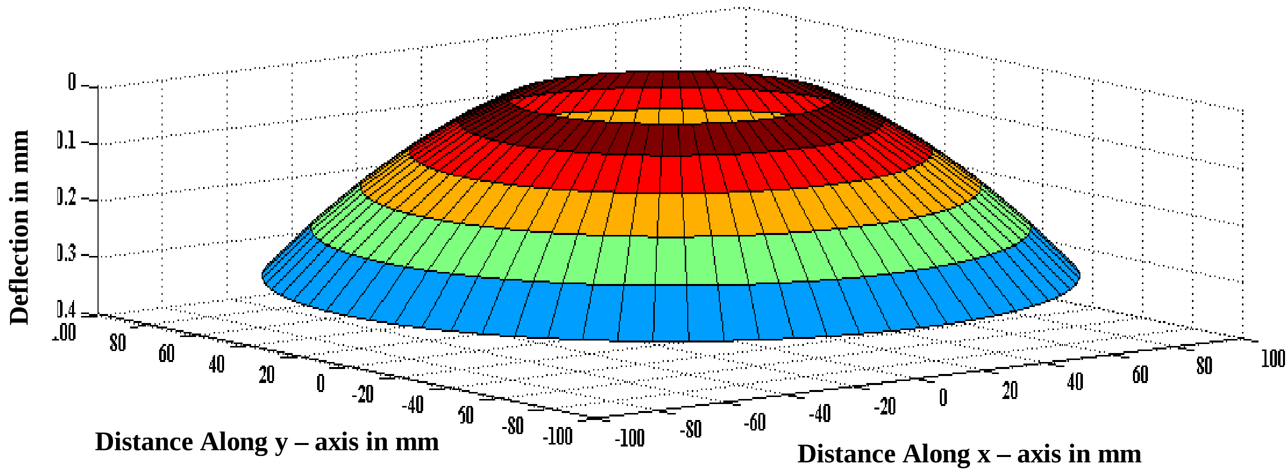


Fig. 5 Deformed shape of annular plate under uniform lateral pressure with clamped inner edge

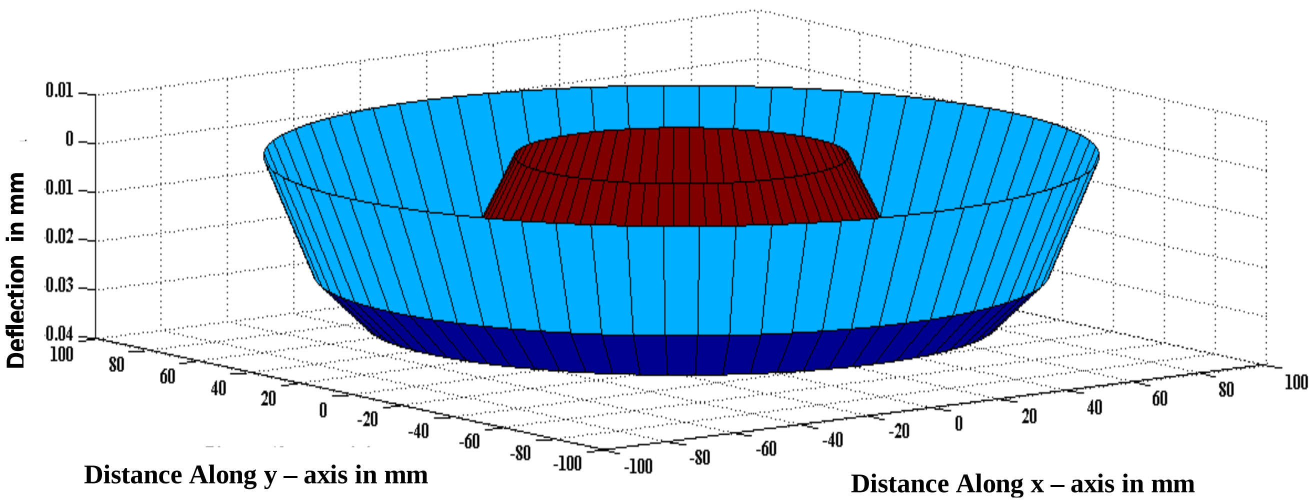


Fig. 6 Deformed shape of annular plate under uniform lateral pressure with inner and outer edges as SS

Table 1 Results Obtained for Clamped Annular Plate

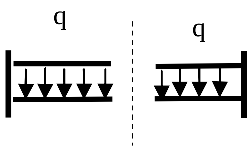
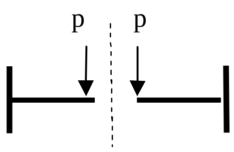
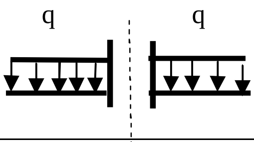
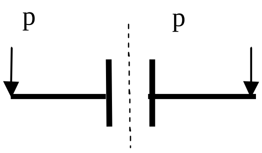
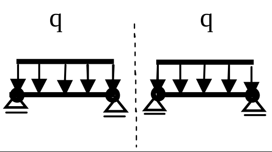
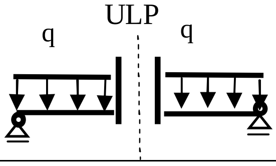
Ex. No.	TYPE	NODE	δ (mm)		Mr (kN-mm)		Mθ (kN-mm)	
			IFM	Exact	IFM	Exact	IFM	Exact
1	Clamped (Outer) ULP 	1	0.0185	0.0199	0.00	0.000	899.14	901.6
		2	0.0081	0.0087	41.32	44.19	596.21	617.3
		3	0.0032	0.0004	89.21	100.54	365.1	394.1
		4	0.0020	0.0003	334.2	339.8	145.12	174.1
		5	0.0001	0.0002	670.4	675.9	59.21	63.11
		6	0.0000	0.0000	998.1	1088	321.14	330.1
2	Clamped (Outer) LL 	1	0.0012	0.002	0.00	0.000	20.14	24.60
		2	0.0014	0.0016	-5.14	-6.01	19.14	24.98
		3	0.0011	0.0013	-9.11	-9.84	19.00	23.66
		4	0.0007	0.0009	-12.00	-12.62	18.11	22.73
		5	0.0003	0.0004	-14.10	-14.81	18.04	21.60
		6	0.000	0.0000	-16.54	-16.58	18.00	20.14
3	Clamped (Inner) ULP 	1	0.000	0.000	-331.2	-335.7	-92.14	-100.7
		2	0.032	0.040	-180.0	-183.3	-214.1	-238.1
		3	0.101	0.140	345.2	375.0	-294.1	-295.8
		4	0.135	0.166	365.1	375.9	-349.1	-357.6
		5	0.211	0.243	228.2	241.86	-442.5	-449.3
		6	0.322	0.332	0.00	0.00	-568.1	-579.8
4	Clamped (Inner) LL 	1	0.000	0.00	-50.00	-57.45	-89.12	-92.45
		2	0.0011	0.0014	-30.44	-34.69	-60.75	-68.69
		3	0.0031	0.0033	-19.45	-21.27	-50.14	-56.27
		4	0.0051	0.0055	-11.47	-12.14	-43.14	-47.14
		5	0.0070	0.008	-5.00	-5.36	-38.54	-40.36
		6	0.091	0.108	0.00	0.00	-30.11	-35.14

Table 2 Results Obtained for Annular Plates Simply Supported at Outer Edge

Ex. No.	TYPE	NODE	δ (mm)		Mr (kN-mm)		Mθ (kN-mm)	
			IFM	Exact	IFM	Exact	IFM	Exact
1	SS (Outer & Inner) ULP 	1	0.00	0.00	0.00	0.00	566.2	597.1
		2	0.018	0.020	-124.3	-127.1	145.3	150.2
		3	0.031	0.032	-345.1	-350.1	-187.2	-195.7
		4	0.030	0.034	-487.1	-500.8	-502.5	-515.8
		5	0.021	0.024	-700.3	-724.1	-823.5	-837.3
		6	0.00	0.00	0.00	0.00	-1021	-1133
2	SS (Outer) & Clamped (Inner) ULP 	1	0.00	0.00	1698	1761	498.2	528.1
		2	0.005	0.0052	534.3	567.1	434.1	471.1
		3	0.011	0.014	-312.1	-339.1	302.3	334.5
		4	0.014	0.020	-1093	-1141	-267.2	-285.1
		5	0.014	0.017	-1832	-1894	-734.1	-741.1
		6	0.00	0.00	-2477	-2652	-1178	-1219